数理解析研究所講究録
1194巻 2001年 241-249

ON A MARKSTART RENDEZVOUS SEARCH[1]

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1. Introduction.

Rendezvous search concerns problems in which two or more people are attempting to meet. Although Schelling had discussed this type of problem in his book [11] in 1960, it was not until 1995 that rendezvous search was put in a rigorous mathematical framework by Alpern [1]. In this framework it is natural to ask not only if the players can meet but also the least time that they can do so; this time is called the rendezvous value of the problem. Alpern's paper has created considerable interest and papers relating to it have now been published. Many of these papers involve two players and, when this happens, the problems can be divided into two cases, the asymmetric in which the players are distinguishable and so can adopt different strategies, and the symmetric in which the players are indistinguishable and forced to use the same strategy.

Very recently Baston and Gal [7] have considered a form of rendezvous search in which a player can recognize another player's starting point when he reaches it. They called this new form markstart rendezvous search to distinguish it from the previous form which they termed classical rendezvous search. Although both forms have connections with linear search, the results in [7] suggest that the forms have different characteristics. Generally speaking, in classical rendezvous search, symmetric problems have so far been considerably more difficult to solve than asymmetric ones. For instance, Alpern and Gal [4] have proved that, when two players are placed at a known distance D apart on the line, the asymmetric rendezvous value is 13D/8 whereas only bounds have been obtained for the corresponding symmetric value (see [4]). In contrast, the results in [7] suggest that, in markstart rendezvous search, symmetric problems appear to be more amenable to analysis than asymmetric ones.

The primary purpose of this paper is to prove two results in markstart rendezvous search. Firstly we show that, if the initial distance of the players on the line is given by the uniform distribution on [0,1], then the (pure) symmetric markstart rendezvous search value is 9/8. Not only does this solve a problem left open in [7] but it also gives a better upper bound than Theorem 4.1 in [7] for the corresponding asymmetric markstart problem. Secondly we demonstrate that there are markstart optimal strategies which oscillate infinitely but **not** around the player's starting point. This is a new type of optimal behaviour for rendezvous search problems and it occurs in ones in which the initial distance is given by a distribution function F which has finite support and for which the probability of being near to $min\{x : F(x) = 1\}$ is very small.

2. Model.

We investigate the symmetric markstart rendezvous search problem on the line in which the initial distance between the players is chosen by the uniform distribution on [0,1] and the players do not have a common notion of positive direction. We discretize the problem by

[1] This research was partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan,No.11680449.

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considering [0,1] to be comprised of discrete points $0, 1/4n, 2/4n, \ldots, 4n/4n$ and consider that player 1 starts at 0 facing in the positive direction and that player 2 is placed equiprobably at the points $\pm 2/4n, \pm 4/4n, \ldots, \pm 4n/4n$ facing equiprobably in the positive and negative directions. Time is assumed to be discrete and, at each instant of time, the players must move to an adjacent point. This means that, at any time, player 1 and player 2 are either both at even points (i.e. of the form $2r/4n$) or both at odd points so that the players cannot pass each other without meeting. It is convenient to think of player 2 comprising of four agents which are placed at $2r/4n$ facing in the positive direction, at $2r/4n$ facing in the negative direction, $-2r/4n$ facing in the positive direction and $-2r/4n$ facing in the negative direction when $2r/4n$ has been chosen via the probability. Such agents will be denoted by a_{+P}, a_{+N}, a_{-P} and a_{-N} respectively. Note that player 1 will meet $a_{+N}(r)$ and $a_{-N}(r)$ before he or the agent finds the starting point of the other whereas player 1 will meet a_{+P} and a_{-P} only after player 1 finds the agent's starting point or the agent finds player 1's starting point. Namely, if player 1 and player 2 are facing in opposite directions then, under a symmetric strategy, they reach each other's starting points at the same time and so they must meet before either player finds the other's starting point. On the other hand, if player 1 and player 2 are facing in the same direction then, under a symmetric strategy, they will remain at their initial distance apart at least until one of them locates the other's starting point.

Since the problem is symmetric, the players use the same strategy which we can consider to be of the form $k_1F\ k_2B\ k_3F\ k_4B, \ldots$ which is interpreted as move k_1 points forward, then k_2 points backward, then k_3 points forward and so on provided the other player's starting point has not been located; once a player locates the other player's starting point he always continues in the direction he was taking when he found the starting point. In calculating when player 1 and an agent meet, it is convenient to have an expression for the position of a player at a particular time under a given strategy s. To obtain this expression we introduce the following notation. Let s be a given strategy, then, for non-negative integers m, put

$$\sigma_s(m) = \sum_{i=1}^{m} k_i \quad \text{and} \quad 4n\delta_s(m) = \sum_{i=1}^{m} (-1)^{i+1} k_i$$

with the convention that $\sigma_s(0) = 0 = \delta_s(0)$; note that $\sigma_s(m)$ is the time and $\delta_s(m)$ is the (signed) distance from his starting point when the player makes his m-th change of direction under s. Now define $g_s(t)$ by

$$4ng_s(t) = 4n\delta_s(i) + (-1)^i (t - \sigma_s(i)) \quad \text{for} \quad \sigma_s(i) \le t \le \sigma_s(i+1).$$

It is easy to check that, if the players have not met and player 1 has not located player 2's starting point, then, under s, player 1 is at $g_s(t)$ at time t. Similarly, if the players have not met and the agent has not located player 1's starting point, then, under s, the agents $a_{+P}(r)$, $a_{+N}(r), a_{-P}(r)$ and $a_{-N}(r)$ will be in the positions

$$g_s(t) + 2r/4n, \quad -g_s(t) + 2r/4n, \quad g_s(t) - 2r/4n, \quad \text{and} \quad -g_s(t) - 2r/4n$$

respectively at time t.

Thus, under s, the meeting times $\tau_{+N}(r; s)$ and $\tau_{-N}(r; s)$ of player 1 with $a_{+N}(r)$ and $a_{-N}(r)$ are the least times t such that $g_s(t) = -g_s(t) + 2r/4n$ and $g_s(t) = -g_s(t) - 2r/4n$ respectively. Thus

$$\tau_{+N}(r; s) = min\{t : g_s(t) = r/4n\} \quad \text{and} \quad \tau_{-N}(r; s) = min\{t : g_s(t) = -r/4n\}.$$

Let

$$\mathcal{A}_+(s) = \{r : g_s(\bar{t}(r;s)) = 2r/4n\} \quad \text{and} \quad \mathcal{A}_-(s) = \{r : g_s(\bar{t}(r;s)) = -2r/4n\}$$

where

$$\bar{t}(r;s) = min\{t : |g_s(t)| = 2r/4n\}.$$

If $r \in \mathcal{A}_+(s)$, then, under s, player 1 locates the starting point of $a_{+P}(r)$ and $a_{-P}(r)$ locates the starting point of player 1; in this case the player or agent locating the starting point continues moving forward. If $r \in \mathcal{A}_-(s)$, then, under s, player 1 locates the starting point of $a_{-P}(r)$ and $a_{+P}(r)$ locates the starting point of player 1; in this case the player or agent locating the starting point continues moving backwards.

It is easy to check that, under s, the meeting times $\tau_{+P}(r;s)$ and $\tau_{-P}(r;s)$ of player 1 with $a_{+P}(r)$ and $a_{-P}(r)$ respectively satisfy $\tau_{+P}(r;s) = \tau_{-P}(r;s) = \tau_P(r;s)$ say and that

$$\tau_P(r;s) = min\{t > \bar{t}(r;s) : t - \bar{t}(r;s) = 4ng_s(t)\} \quad \text{if} \quad r \in \mathcal{A}_+(s)$$

and

$$\tau_P(r;s) = min\{t > \bar{t}(r;s) : t - \bar{t}(r;s) = -4ng_s(t)\} \quad \text{if} \quad r \in \mathcal{A}_-(s).$$

The expected meeting time $E(s)$ of the players under s is then given by

$$E(s) = \sum_{r=1}^{2n} (\tau_{+N}(r) + \tau_{-N}(r) + 2\tau_P(r))/8n.$$

Remark 2.1 Let s be a strategy and $0 < r \leq 2n$.

(i) If $r \in \mathcal{A}_+(s)$, then $g_s(\tau_P(r;s)) > 0$ and $\tau_P(r;s) \in [\sigma_s(2z-1)+1, \sigma_s(2z)]$ for some z.

(ii) If $r \in \mathcal{A}_-(s)$, then $g_s(\tau_P(r;s)) < 0$ and $\tau_P(r;s) \in [\sigma_s(2z)+1, \sigma_s(2z+1)]$ for some z.

Proof. Immediate from the definitions on noting that g_s is decreasing in $[\sigma_s(2z-1), \sigma_s(2z)]$ and g_s is increasing in $[\sigma_s(2z), \sigma_s(2z+1)]$.

Remark 2.2 Let s be a strategy and $0 < r \leq 2n$.

(i) If $r \in \mathcal{A}_+(s)$ and $r \leq 2n\delta_s(2z-1) \leq k_{2z}$, then $\tau_P(r;s) \leq \sigma_s(2z-1) + 2n\delta_s(2z-1)$.

(ii) If $r \in \mathcal{A}_-(s)$ and $r \leq -2n\delta_s(2z) \leq k_{2z+1}$, then $\tau_P(r;s) \leq \sigma_s(2z) - 2n\delta_s(2z)$.

Proof. We will prove only (i) because (ii) follows similarly. Suppose $r \in \mathcal{A}_+(s)$ and $r \leq 2n\delta_s(2z-1) \leq k_{2z}$, then $g_s(0) = 0 < 2r \leq 4ng_s(\sigma_s(2z-1))$ gives $\bar{t}(r;s) \leq \sigma_s(2z-1)$ because $g_s(t+1) - g_s(t) \in \{1, -1\}$. Put $f_s(t) = t - \bar{t}(r;s) - 4ng_s(t)$ then $f_s(\bar{t}(r;s)) = -2r < 0$ and $f_s(\sigma_s(2z-1) + 2n\delta_s(2z-1)) \geq 2n\delta_s(2z-1) - g_s(\sigma_s(2z-1) + 2n\delta_s(2z-1)) \geq 0$ because $k_{2z} \geq 2n\delta_s(2z-1)$. Now $f_s(t+1) - f_s(t) \in \{0, 2, -2\}$ so $f_s(t)$ is even for $t \geq \bar{t}(r;s)$. Hence there is a t^* satisfying $\bar{t}(r;s) < t^* \leq \sigma_s(2z-1) + 2n\delta_s(2z-1)$ such that $f_s(t^*) = 0$. Thus $\tau_P(r;s) \leq t^*$ and the result follows.

3. Analysis and Results.

In this section we give the main theorem which gives a solution for the pure symmetric markstart rendezvous search model. The proofs of the lemmas use a contradiction argument and most follow the same pattern. They assume that there is an optimal strategy s^* satisfying certain conditions and then show that there is another strategy s which is an improvement on s^*. In these proofs we will always assume without stating it that s^* is given by $h_1F\ h_2B\ h_3F\ h_4B\ldots$ and s by $k_1F\ k_2B\ k_3F\ k_4B\ldots$. In addition we assume that the form $h_1F\ h_2B\ldots$ is finite with $h_i > 0$ and that M is the least M for which all agents are met by player 1 on or before time $\sigma_{s^*}(M)$. To avoid awkward special conditions in the statements of some lemmas, we adopt the convention that $h_M = 10n$.

If the players have not met and a player has not located the other player's starting point by the time he reaches a point at distance one from his own starting point, O say, he knows that the other's starting point lies on the other side of O and so will start to move back towards O. Thus a player moves a distance at most one from his starting point except possibly when he has changed direction the full number (M) of times. This is the import of our first result.

Lemma 1. Let s^* be an optimal strategy, then $|\delta_{s^*}(w)| \le 1$ if $w < M$.

Proof. Suppose s^* is optimal with $w < M$ and $|4n\delta_{s^*}(w)| = 4n + \alpha$ where $\alpha \ge 1$. We prove only the case $4n\delta_{s^*}(w) = 4n + \alpha$ because the other case follows similarly. Let $4n\delta_{s^*}(w) = 4n + \alpha$, then we can take w to be odd. Consider the strategy s with $k_i = h_i$ for $i \le w - 1$, $k_w = h_w - 1$ and $k_{w+1} = 10n + \alpha$. Note that, for all r, $\bar{t}(r; s^*) \ne \sigma_{s^*}(w)$ and $\tau_\beta(r; s^*) \ne \sigma_{s^*}(w)$ for $\beta \in \{P, +N, -N\}$. Clearly $g_s(t) = g_{s^*}(t)$ for $t < \sigma_{s^*}(w) - 1$ so $\tau_\beta(r; s) = \tau_\beta(r; s^*)$ if $\tau_\beta(r; s^*) \le \sigma_{s^*}(w) - 1$. Further, since $g_s(t) < g_{s^*}(t)$ for $t \ge \sigma_s(w)$, $\tau_\beta(r; s) < \tau_\beta(r; s^*)$ for $\beta \in \{P, -N\}$ when $\tau_\beta(r; s^*) > \sigma_{s^*}(w)$; there are such r because $\tau_{+N}(r; s^*) < \sigma_{s^*}(w)$ for all r and $w < M$ so $E(s) < E(s^*)$ which contradicts the optimality of s^* and the lemma follows. ∎

Lemma 2. Let s^* be an optimal strategy.

(a) (i) If $\delta_{s^*}(2z+1) > 0$, then $\delta_{s^*}(2i+1) < \delta_{s^*}(2z+1)$ for $i < z$.

(ii) If $\delta_{s^*}(2z) < 0$, then $\delta_{s^*}(2i) > \delta_{s^*}(2z)$ for $i < z$.

(b) If $|\delta_{s^*}(i)| \ge 1/2$, then $4n\delta_{s^*}(i)$ is even.

Proof. The proofs of (a) and (b) use the same modified strategy to obtain a contradiction and so we will start the proofs of (a) and (b) in (A) and (B) below and complete them in (C) below.

(A) The proof of (ii) is similar to that of (i) so we will only prove (i). Suppose the result is false then there is an optimal strategy s^* with $\delta_{s^*}(2z+1) > 0$ and a $j < z$ satisfying $\delta_{s^*}(2j+1) \ge \delta_{s^*}(2z+1)$.

(B) Suppose s^* is an optimal strategy for which there is a w such that $|\delta_{s^*}(w)| \ge 1/2$ and $4n\delta_{s^*}(w)$ is odd. We prove only the case when w is odd, say $2z+1$, because the other case follows similarly.

(C) Clearly, in both cases (A) and (B), there is no r such that any of $\tau_{+N}(r; s^*)$, $\tau_{-N}(r; s^*)$ and $\bar{t}(r; s^*)$ are in $[\sigma_{s^*}(2z+1), \sigma_{s^*}(2z+1)+1]$. Furthermore, using Remark 2.1, $\tau_P(r; s^*) \ne \sigma_{s^*}(2z+1)$ for any r.

Let w' denote the least value of $w > z$ for which there is an $r \le \max_{i \le z} \delta_{s^*}(2i+1)$ such that $\tau_P(r; s^*) = \sigma_{s^*}(2w)$; if no such w exists we take $w = \infty$. Now consider the strategy s given by $k_i = h_i$ for $i \le 2z$ and $i > 2w' + 1$, $k_{2z+1} = h_{2z+1} - 1$, $k_{2w'+1} = h_{2w'+1} + 1$; if $w' = z + 1$, $k_{2w'} = h_{2w'}$ whereas, if $w' > z + 1$, $k_{2z+2} = h_{2z+2} - 1$, $h_i = k_i$ for $2z + 3 \le i < 2w'$ and $k_{2w'} = h_{2w'} + 1$.

Note that, if $w' \neq \infty$, it follows from Remarks 2.1 and 2.2 that, for any $r \in \mathcal{A}_-(s^*)$ with $\bar{t}(r; s^*) < \sigma_{s^*}(2z+1)$, we have $\tau_P(r; s^*) < \sigma_{s^*}(2z+1)$ and also that $g_{s^*}(t) > 0$ for $\sigma_{s^*}(2z+1) < t \leq \sigma_{s^*}(2w')$ so that, in particular, $\bar{t}(r; s^*) \neq \sigma_{s^*}(2w')$. We have

$$\sigma_s(i) = \sigma_{s^*}(i) \quad \text{and} \quad \delta_s(i) = \delta_{s^*}(i) \quad \text{for} \quad i \leq 2z,$$

$$\sigma_s(2z+1) = \sigma_{s^*}(2z+1) - 1 \quad \text{and} \quad \delta_s(2z+1) = \delta_{s^*}(2z+1) - 1/4n$$

$$\sigma_s(2w') = \sigma_{s^*}(2w') - 1 \quad \text{and} \quad \delta_s(2w') = \delta_{s^*}(2w') - 1/4n$$

$$\sigma_s(i) = \sigma_{s^*}(i) \quad \text{and} \quad \delta_s(i) = \delta_{s^*}(i) \quad \text{for} \quad i \geq 2w'+1$$

and, if $w' > z+1$,

$$\sigma_s(i) = \sigma_{s^*}(i) - 2 \quad \text{and} \quad \delta_s(i) = \delta_{s^*}(i) \quad \text{for} \quad 2z+2 \leq i < 2w'.$$

It is routine to check that

$$g_s(t) = \begin{cases} g_{s^*}(t) & \text{if } t \leq \sigma_{s^*}(2z+1) - 1 \text{ or } t \geq \sigma_{s^*}(2w') \\ g_{s^*}(t+2) & \text{if } \sigma_{s^*}(2z+1) \leq t \leq \sigma_{s^*}(2w') - 2 \\ g_{s^*}(\sigma_{s^*}(2w')) - 1/(4n) & \text{if } t = \sigma_{s^*}(2w') - 1 \end{cases} .$$

Hence, for $\beta \in \{+N, -N\}$,

$$\tau_\beta(r; s) \leq \begin{cases} \tau_\beta(r; s^*) - 2 & \text{if } \sigma_{s^*}(2z+1) \leq \tau_\beta(r; s^*) \leq \sigma_{s^*}(2w') - 1 \\ \tau_\beta(r; s^*) & \text{otherwise} \end{cases} .$$

Clearly $\tau_P(r; s) = \tau_P(r; s^*)$ if $\tau_P(r; s^*) \leq \sigma_{s^*}(2z+1) - 1$ or $\bar{t}(r; s^*) > \sigma_{s^*}(2w')$. Further, if $\bar{t}(r; s^*) \leq \sigma_{s^*}(2z+1) < \tau_P(r; s^*) < \sigma_{s^*}(2w')$, then $\tau_P(r; s) \leq \tau_P(r; s^*) - 1$, because

$$\begin{aligned} \tau_P(r; s^*) - 1 - \bar{t}(r; s) &= \tau_P(r; s^*) - 1 - \bar{t}(r; s^*) = 4ng_{s^*}(\tau_P(r; s^*)) - 1 \\ &= \begin{cases} 4ng_{s^*}(\tau_P(r; s^*) + 1) = 4ng_s(\tau_P(r; s^*) - 1) & \text{if } \tau_P(r; s^*) \neq \sigma_{s^*}(2w') \\ 4ng_s(\tau_P(r; s^*) - 1) & \text{if } \tau_P(r; s^*) = \sigma_{s^*}(2w') \end{cases} \end{aligned}$$

If $\bar{t}(r; s^*) \leq \sigma_{s^*}(2z+1)$ and $\sigma_{s^*}(2w') < \tau_P(r; s^*)$, then $\tau_P(r; s) \leq \tau_P(r; s^*)$ because

$$\tau_P(r; s^*) - \bar{t}(r; s) = \tau_P(r; s^*) - \bar{t}(r; s^*) = 4ng_{s^*}(\tau_P(r; s^*)) = 4ng_s(\tau_P(r; s^*)).$$

Further $\tau_P(r; s) \leq \tau_P(r; s^*) - 2$ if $\sigma_{s^*}(2z+1) - 1 < \bar{t}(r; s^*) \leq \tau_P(r; s^*) \leq \sigma_{s^*}(2w')$ because

$$\tau_P(r; s^*) - 2 - \bar{t}(r; s) = \tau_P(r; s^*) - \bar{t}(r; s^*) = 4ng_{s^*}(\tau_P(r; s^*)) = 4ng_s(\tau_P(r; s^*) - 2).$$

If $\sigma_{s^*}(2z+1) - 1 < \bar{t}(r; s^*) < \sigma_{s^*}(2w') < \tau_P(r; s^*)$, then $\tau_P(r : s) \leq \tau_P(r; s^*) - 1$ because

$$\begin{aligned} \tau_P(r; s^*) - 1 - \bar{t}(r; s) &= \tau_P(r; s^*) + 1 - \bar{t}(r; s^*) = 4ng_{s^*}(\tau_P(r; s^*)) + 1 \\ &= 4ng_{s^*}(\tau_P(r; s^*) - 1) = 4ng_s(\tau_P(r; s^*) - 1) \end{aligned}$$

Note that any meeting time of the players between $\sigma_{s^*}(2z+1)$ and $\sigma_{s^*}(2w')$ is strictly better under s than s^* and there is such a meeting for some r. All the other meeting times are at least as good under s than under s^* so $E(s) < E(s^*)$ which is a contradiction and the lemma follows. ∎

Our next result tells us that there is an optimal strategy for the players which oscillates around the player's starting point. Note that this result is not true for probability distributions in general as Theorem shows.

Notation. We will use $\mathcal{S}$ to denote the set of all optimal strategies s^* such that $\delta_{s^*}(2i) < 0$ and $\delta_{s^*}(2i-1) > 0$ for all $i \geq 0$ for which the δ_{s^*} are defined.

Lemma 3. $\mathcal{S} \neq \emptyset$.

Proof. For a strategy s define $q(s)$ by

$$q(s) = |\{i : \delta_s(2i) \geq 0 \quad \text{or} \quad \delta_s(2i-1) \leq 0\}|.$$

Suppose $\mathcal{S} = \emptyset$, then we can choose an optimal strategy s^* such that $q(s^*) = min\{q(s) : s \text{ is optimal}\}$. Now $q(s^*) \geq 1$ so we can take w to be the maximum value of j satisfying $(-1)^{j-1}\delta_{s^*}(j) > 0$. We will consider only the case in which there is a w is even, say $w = 2z$, because the other case follows similarly. Now $2z < M$ because $\delta_{s^*}(2z-1) \leq 1$ by Lemma 1 and $h_{2z} \neq 8n$; hence $\delta_{s^*}(2z+1)$ is well-defined and positive so $\delta_{s^*}(2z+1) > max_{0 \leq i < z}\, \delta_{s^*}(2i+1)$ by Lemma 2 and $h_{2z+1} > h_{2z}$.

Consider the strategy s given by $k_i = h_i$ for $i \leq 2z-2$, $k_{2z-1} = h_{2z-1} + h_{2z+1} - h_{2z}$ and $k_i = h_{i+2}$ for $i \geq 2z$. We have

$$\sigma_s(i) = \sigma_{s^*}(i) \quad \text{and} \quad \delta_s(i) = \delta_{s^*}(i) \quad \text{for} \quad i \leq 2z-2$$

and

$$\sigma_s(i) = \sigma_{s^*}(i+2) - 2h_{2z} \quad \text{and} \quad \delta_s(i) = \delta_{s^*}(i+2) \quad \text{for} \quad i \geq 2z-1.$$

It is easy to check that $g_s(t) = g_{s^*}(t + 2h_{2z})$ for $t \geq \sigma_{s^*}(2z-1) + 2h_{2z}$. Because $\delta_{s^*}(2z) > 0$, none of $\tau_{+N}(r; s^*)$, $\tau_{-N}(r; s^*)$ and $\bar{t}(r; s^*)$ are in $[\sigma_{s^*}(2z-1), \sigma_{s^*}(2z-1) + 2h_{2z}]$ for any r and so it follows that, for all r,

$$\tau_{+N}(r; s) \leq \tau_{+N}(r; s^*), \qquad \tau_{-N}(r; s) \leq \tau_{-N}(r; s^*)$$

and

$$\bar{t}(r; s^*) \leq \sigma_{s^*}(2z-1) \quad \text{or} \quad \bar{t}(r; s^*) \geq \sigma_{s^*}(2z-1) + 2h_{2z}.$$

Hence

$$\bar{t}(r; s) = \begin{cases} \bar{t}(r; s^*) & \text{if } \bar{t}(r; s^*) \leq \sigma_{s^*}(2z-1) \\ \bar{t}(r; s^*) - 2h_{2z} & \text{if } \bar{t}(r; s^*) \geq \sigma_{s^*}(2z-1) + 2h_{2z} \end{cases}.$$

Hence, for any r satisfying $\bar{t}(r; s^*) \geq \sigma_{s^*}(2z-1) + 2h_{2z}$, we have $\tau_P(r; s) \leq \tau_P(r; s^*) - 2h_{2z}$. It is straightforward to check that, for any r such that $\bar{t}(r; s^*) \leq \sigma_{s^*}(2z-1)$ and $g_{s^*}(\bar{t}(r; s^*)) = -2r/4n$, we have $\tau_P(r; s^*) < \sigma_{s^*}(2z-1)$ and so $\tau_P(r; s) = \tau_P(r; s^*)$.

Hence let r^* be such that $\bar{t}(r^*; s^*) \leq \sigma_{s^*}(2z-1)$ and $g_{s^*}(\bar{t}(r^*; s^*)) = 2r/4n$. If $\tau_P(r^*; s^*) \leq \sigma_{s^*}(2z-1)$, $\tau_P(r^*; s) = \tau_P(r^*; s^*)$. We now consider two cases.

(α) Suppose $\tau_P(r^*; s^*) \geq \sigma_{s^*}(2z+1)$, then $\sigma_s(j) \leq \tau_P(r^*; s^*) \leq \sigma_s(j+1)$ for some odd $j \geq 2z+1$ so $\sigma_s(j-2) \leq \tau_P(r^*; s^*) - 2h_{2z} \leq \sigma_s(j-1)$ and we have

$$\begin{aligned} \tau_P(r^*; s^*) - \bar{t}(r^*; s) &= \tau_P(r^*; s^*) - \bar{t}(r^*; s^*) = 4ng_{s^*}(\tau(r^*; s^*)) \\ &= 4ng_{s^*}(\tau(r^*; s) - 2h_{2z}) \end{aligned}$$

Thus

$$\begin{aligned}\tau_P(r^*;s^*) + h_{2z} - \bar{t}(r^*;s) &= 4ng_{s^*}(\tau(r^*;s^*) - 2h_{2z}) + h_{2z}\\ &\geq 4ng_{s^*}(\tau(r^*;s) - h_{2z})\end{aligned}$$

so $\tau_P(r^*;s) \leq \tau_P(r^*;s^*) + h_{2z}$.

(β) Suppose $\sigma_{s^*}(2z-1) < \tau_P(r^*;s^*) < \sigma_{s^*}(2z+1)$. Because $g_{s^*}(\bar{t}(r^*;s^*)) = 2r/4n$, we must have $\sigma_{s^*}(2z-1) < \tau_P(r^*;s^*) \leq \sigma_{s^*}(2z)$ so

$$\begin{aligned}\sigma_s(2z-1) = \sigma_{s^*}(2z-1) + h_{2z+1} - h_{2z} &\leq \tau_P(r^*;s^*) + h_{2z+1} - h_{2z}\\ &\leq \sigma_{s^*}(2z) + h_{2z+1} - h_{2z} = \sigma_s(2z-1) + h_{2z} \leq \sigma_s(2z)\end{aligned}$$

because, by the maximality of z and Lemma 2,

$$h_{2z+2} \geq 4n\delta_{s^*}(2z+1) > 4n\delta_{s^*}(2z-1) > h_{2z}.$$

We therefore have

$$\begin{aligned}\tau_P(r^*;s^*) + h_{2z+1} - h_{2z} - \bar{t}(r^*;s) &= \tau_P(r^*;s^*) + h_{2z+1} - h_{2z} - \bar{t}(r^*;s^*)\\ &= 4ng_{s^*}(\tau_P(r^*;s^*)) + h_{2z+1} - h_{2z}\\ &= 4n\delta_{s^*}(2z-1) - (\tau_P(r^*;s^*) - \sigma_{s^*}(2z-1)) + h_{2z+1} - h_{2z}\\ &= 4n\delta_s(2z-1) - (\tau_P(r^*;s^*) + h_{2z+1} - h_{2z} - \sigma_s(2z-1))\\ &= 4ng_s(\tau_P(r^*;s^*) + h_{2z+1} - h_{2z})\end{aligned}$$

so

$$\tau_P(r^*;s) \leq \tau_P(r^*;s^*) + h_{2z+1} - h_{2z} \tag{1}$$

Since $\tau_P(r_1) \neq \tau_P(r_2)$ for $r_1 \neq r_2$, there are at most h_{2z} values of r satisfying $\tau_P(r;s^*) \in [\sigma_{s^*}(2z-1)+1, \sigma_{s^*}(2z)]$ and so a total of at most $2h_{2z}$ agents for which (1) holds.

If there are no r^* for which (1) holds, $E(s) < E(s^*)$. Hence we may assume that there are r^* for which (1) holds and then $2r^* \leq 4n\delta_{s^*}(2z-1)$ and $\bar{t}(r;s^*) > \sigma_{s^*}(2z-1)$ for $2r > 4n\delta_{s^*}(2z-1)$. But then there are at least $\lceil(4n - 4n\delta_{s^*}(2z-1))/2\rceil \geq \lceil(h_{2z+1} - h_{2z})/2\rceil$ values of r for which $\bar{t}(r;s^*) > \sigma_{s^*}(2z-1)$ and therefore at least $h_{2z+1} - h_{2z}$ agents for which $\tau_P(r;s) \leq \tau_P(r;s^*) - 2h_{2z}$.

Hence $E(s) \leq E(s^*)$ so that s is also optimal. But $q(s) < q(s^*)$ which gives a contradiction and the lemma follows. ∎

Intuitively it is fairly obvious that, if an optimal strategy oscillates about the player's starting point, then the oscillations will get bigger and bigger. Our next lemma proves that this intuition is correct. We omit proofs of Lemmas 4-6. See [8] for those.

Lemma 4. If $s^* \in \mathcal{S}$, then $|\delta_{s^*}(i)|$ for $i = 1, \ldots, M$ is a strictly increasing sequence.

Lemma 5. Let $s^* \in \mathcal{S}$, then $|\delta_{s^*}(i)| \geq 1/2$ implies $|\delta_{s^*}(i+1)| = 1$.

Lemma 6. Let $s^* \in \mathcal{S}$, then $\delta_{s^*}(1) < 1/2$ implies $\delta_{s^*}(2) \leq -1/2$.

Theorem 7. The pure symmetric markstart rendezvous value for the uniform distribution on [0,1] is $(9n+1)/8n$ and an optimal strategy for the players is to move forward for 1/2, then backwards for 3/2 and then forwards thereafter.

Proof. By Lemma 3 there is an optimal strategy in $\mathcal{O}$. We divide the analysis into two cases.

(i) First assume $h_1 \geq 2n$, then $\delta_{s^*}(2) = -1$ by Lemma 5 and it is easy to see that $M = 2$ if $h_1 = 4n$ and $M = 3$ otherwise; in the latter case $h_2 = h_1 + 4n$ and $h_3 = 8n$. We have

$$(4n)(8n)E(s^*) = \sum_{r=1}^{2n} r \; + 2\sum r = 1^{\lfloor h_1/2 \rfloor}(h_1 + r) \; + 2 \sum_{r=\lfloor h_1/2 \rfloor+1}^{2n} (2h_1 + 4n + r) \; + \sum_{r=1}^{2n}(2h_1 + r)$$
$$= 4\sum_{r=1}^{2n} r \; + 2h_1(6n - \lfloor h_1/2 \rfloor) \; + 8n(2n - \lfloor h_1/2 \rfloor).$$

Whether $h_1$is odd or even, the expression is a negative quadratic in h_1 and so has a minimum at one of the extreme points of the range for h_1. Since $h_1 \in [2n, 4n]$, we have

$$32n^2 E(s^*) = 4n(2n+1) + 16n^2 + min\{4n(5n) - 8n^2, 8n(4n) - 16n^2\}$$
$$= 36n^2 + 4n,$$

the minimum being achieved when $h_1 = 2n$.

(ii) Now assume $h_1 < 2n$, then $\delta_{s^*}(2) \leq -1/2$ by Lemma 6 and then $-4n\delta_{s^*}(2) = h_2 - h_1$ is even by Lemma 2(b) and $\delta_{s^*}(3) = 1$ by Lemma 5. Hence $M = 3$ if $\delta_{s^*}(2) = -1$ and $M = 4$ otherwise; in the latter case, $h_3 = h_2 - h_1 + 4n$ and $h_4 = 8n$. We have

$$(4n)(8n)E(s^*) = \sum_{r=1}^{h_1} r \; + \sum_{h_1+1}^{2n}(2h_2 - h_1 + r) \; + 2\sum_{r=1}^{\lfloor h_1/2 \rfloor}(h_1 + r)$$
$$+ 2\sum_{r=\lfloor h_1/2 \rfloor+1}^{(h_2-h_1)/2}(h_1 + h_2 + r) + 2\sum_{r=1+(h_2-h_1)/2}(2h_2 + 4n + r) \; + \sum_{r=1}^{2n}(2h_1 + r)$$
$$= 4\sum_{r=1}^{2n} r \; + (2h_2 - h_1)(2n - h_1) + 2h_1((h_2 - h_1)/2)$$
$$+ 2h_2(2n - \lfloor h_1/2 \rfloor + 2n - (h_2 - h_1)/2) + 8n(2n - (h_2 - h_1)/2) + 2h_1 2n.$$

The expression is a negative quadratic in h_2 and so has its minimum at one of the extreme points of the range of h_2. By Lemma 6 $h_2 - h_1 \geq 2n$ so, because $h_1 < 2n$, the extremes are $h_1 + 2n$ and $h_1 + 4n$. Routine calculations show that the minimizing point is the former if h_1 satisfies $h_1 + \lfloor h_1/2 \rfloor < n$ and the latter otherwise.

For the first case substituting $h_2 = h_1 + 2n$ into $E(s^*)$ gives

$$(4n)(8n)E(s^*) = 36n^2 + 4n + 10nh_1 - h_1^2 - 2(h_1 + 2n)\lfloor h_1/2 \rfloor \geq 36n^2 + 4n + 2h_1(4n - h_1)$$
$$> n36n^2 + 4n$$

in the range under consideration.

For the second case substituting $h_2 = h_1 + 4n$ into $E(s^*)$ gives

$$(4n)(8n)E(s^*) = 40n^2 + 4n - h_1^2 - 2(h_1 + 4n)\lfloor h_1/2 \rfloor + 6nh_1$$
$$\geq 40n^2 + 4n - 2h_1^2 + 2nh_1 = 40n^2 + 4n + 2h_1(n - h_1)$$
$$\geq 36n^2 + 4n$$

in the range under consideration and the result follows. ∎

References.

[1] Alpern,S.:The Rendezvous Search Problem. *SIAM J. on Control and Optimization* **33** (1995), 673-683.

[2] Alpern,S. Baston,V. and Essegaier,S.: Rendezvous Search on a Graph. *J. Applied Probability* **36** (1999), 223-231.

[3] Alpern,S. and Beck,A.: Asymmetric Rendezvous on the Line is a Double Linear Search Problem. *Mathematics of Operations Research* **24** (1999),604-618.

[4] Alpern,S. and Gal,S.:The Rendezvous Search Problem on the Line with Distinguishable Players. *SIAM J. on Control and Optimization* **33**(1995), 1270-76.

[5] Anderson, E.J. and Essegaier,S.: Rendezvous Search on the Line with Distinguished Players. *SIAM J. on Control and Optimization* **33** (1995), 1637-42.

[6] Baston,V.: Two Rendezvous Search Problems on the Line. *Naval Research Logistics* **46**(1999) 335-340.

[7] Baston,V. and Gal,S. mimeo. 2000.

[8] Baston, V. and Kikuta,K. mimeo. 2000.

[9] Lim, W.S. and Alpern, S.: Minimax Rendezvous Search on the Line. *SIAM J. on Control and Optimization* **34** (1996), 1650-65.

[10] Lim, W.S.,Alpern, S. and Beck,A.: Rendezvous Search on the Line with More Than Two Players. *Operations Research* **45**(1997), 357-364.

[11] Schelling,T.C.: *The Strategy of Conflict.* Harvard Univ.Press,1960.